

# Quantum Models for Psychological Measurements: An Unsolved Problem

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## Abstract

There has been a strong recent interest in applying quantum mechanics (QM) outside physics, including in cognitive science. We analyze the applicability of QM to two basic properties in opinion polling. The first property (response replicability) is that, for a large class of questions, a response to a given question is expected to be repeated if the question is posed again, irrespective of whether another question is asked and answered in between. The second property (question order effect) is that the response probabilities frequently depend on the order in which the questions are asked. Whenever these two properties occur together, it poses a problem for QM. The conventional QM with Hermitian operators can handle response replicability, but only in the way incompatible with the question order effect. In the generalization of QM known as theory of positive-operator-valued measures (POVMs), in order to account for response replicability, the POVMs involved must be conventional operators. Although these problems are not unique to QM and also challenge conventional cognitive theories, they stand out as important unresolved problems for the application of QM to cognition. Either some new principles are needed to determine the bounds of applicability of QM to cognition, or quantum formalisms more general than POVMs are needed.

KEYWORDS: decision making, opinion polling, psychophysics, quantum cognition, quantum mechanics, question order effect, response replicability, sequential effects.

## 1 Introduction

Quantum mechanics (QM) is the mathematical formalism of quantum physics. (Sometimes the two are considered synonymous, in which case what we call here QM would have to be called “mathematical formalism of QM.”) However, QM has recently begun to be used in various domains outside of physics, e.g., in biology and economics [1]–[6], as well as in cognitive science [7]–[25]. See recently published monographs [17] and [5] for overviews, as well as the recent target article in *Brain and Behavioral Sciences* [22] with ensuing commentaries and rejoinders. There is one obvious similarity between cognitive science and quantum physics: both deal with observations that are fundamentally probabilistic. This similarity

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makes the use of QM in cognitive science plausible, as QM is specifically designed to deal with random variables. Here, we analyze the applicability of QM in opinion-polling, and compare it to psychophysical judgments.

On a very general level, QM accounts for the probability distributions of measurement results using two kinds of entities, called *observables*  $A$  and *states*  $\psi$  (of the system on which the measurements are made). Let us assume that measurements are performed in a series of consecutive trials numbered  $1, 2, \dots$ . In each trial  $t$  the experimenter decides what measurement to make (e.g., what question to ask), and this amounts to choosing an observable  $A$ . Despite its name, the latter is not observable per se, in the colloquial sense of the word, but it is associated with a certain set of values  $v(A)$ , which are the possible results one can observe by measuring  $A$ . In a psychological experiment these are the responses that a participant is allowed to give, such as *Yes* and *No*.

The probabilities of these outcomes in trial  $t$  (conditioned on all the previous measurements and their outcomes) are computed as some function of the observable  $A$  and of the state  $\psi^{(t)}$  in which the system (a particle in quantum physics, or a participant in psychology) is at the beginning of trial  $t$ ,

$$\Pr[v(A) = v \text{ in trial } t \mid \text{measurements in trials } 1, \dots, t-1] = F(\psi^{(t)}, A, v). \quad (1)$$

This measurement changes the state of the system, so that at the end of trial  $t$  the state is  $\psi^{(t+1)}$ , generally different from  $\psi^{(t)}$ . The change  $\psi^{(t)} \rightarrow \psi^{(t+1)}$  depends on the observable  $A$ , the state  $\psi^{(t)}$ , and the value  $v(A)$  observed in trial  $t$ ,

$$\psi^{(t+1)} = G(\psi^{(t)}, A, v). \quad (2)$$

On this level of generality, a psychologist will easily recognize in (1)-(2) a probabilistic version of the time-honored Stimulus-Organism-Response (S-O-R) scheme for explaining behavior [26]. This scheme involves stimuli (corresponding to  $A$ ), responses (corresponding to  $v$ ), and internal states (corresponding to  $\psi$ ). It does not matter whether one simply identifies  $A$  with a stimulus, or interprets  $A$  as a kind of internal representation thereof, while interpreting the stimulus itself as part of the measurement procedure (together with the instructions and experimental set-up, that are usually fixed for the entire sequence of trials). What is important is that the stimulus determines the observable  $A$  uniquely, so that if the same stimulus is presented in two different trials  $t$  and  $t'$ , one can assume that  $A$  is the same in both of them.<sup>1</sup>

The state  $\psi^{(t+1)}$  determined by (2) may remain unchanged between the response  $v$  terminating trial  $t$  and the presentation of (the stimulus corresponding to) the new observable that initiates trial  $t+1$ . In some applications this interval can indeed be negligibly small or even zero, but if it is not, one has to allow for the evolution of  $\psi^{(t+1)}$  within it. In QM, the “pure” evolution of the state (assuming no intervening inter-trial inputs) is described by some function

$$\psi_{\Delta}^{(t+1)} = H(\psi^{(t+1)}, \Delta), \quad (3)$$

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<sup>1</sup>This approach is adopted here to unify it formally with psychological measurements and quantum measurements in physics (such as spin measurements, mentioned below). However, one of the authors (JRB) prefers another approach, adopted in Ref. [17], in which stimuli are mapped into different states and the observable is fixed by the question about the stimuli. The two approaches coincide with regard to the measurement issues addressed in this article, and therefore the analysis is not affected. The relation between the two approaches is outside the scope of this paper.

where  $\Delta$  is the time interval between the recording of  $v$  in trial  $t$  and the observable in trial  $t + 1$ . This scheme is somewhat simplistic: one could allow  $H$  to depend, in addition to the time interval  $\Delta$ , on the observable  $A$  and the outcome  $v$  in trial  $t$ . We do not consider such complex inter-trial dynamics schemes in this paper.

The reason we single out opinion-polling and compare it to psychophysics is that they exemplify two very different types of stimulus-response relations.

In a typical opinion-polling experiment, a group of participants is asked one question at a time, e.g.,  $a$  = “Is Bill Clinton honest and trustworthy?” and  $b$  = “Is Al Gore honest and trustworthy?” [27]. The two questions, obviously, differ from each other in many respects, none of which has anything to do with their content: the words “Clinton” and “Gore” sound different, and the participants know many aspects in which Clinton and Gore differ, besides their honesty or dishonesty. Therefore, if a question, say,  $b$ , were presented to a participant more than once, she would normally recognize that it had already been asked, which in turn would compel her to repeat it, unless she wants to contradict herself. One can think of situations when the respondent can change her opinion, e.g., if another question posed between two replications of the question provides new information or reminds something forgotten. Thus, if the answer to the question  $a$  = “Do you want to eat this chocolate bar?” is Yes, and the second question is  $b$  = “Do you want to lose weight?,” the replications of  $a$  may very well elicit response No. It is even conceivable that if one simply repeats the chocolate question twice, the person will change her mind, as she may think the replication of the question is intended to make her “think again.” In a wide class of situations, however, changing one’s response would be highly unexpected and even bizarre (e.g., replace  $a$  in the example above with “Do you like chocolate?”). We assume that the pairs of questions asked, e.g., in Moore’s study [27] are of this type.

In a typical psychophysical task, the stimuli used are identical in all respects except for the property that a participant is asked to judge. Consider a simple detection paradigm in which the observer is presented one stimulus at a time, the stimulus being either  $a$  (containing a signal to be detected) or  $b$  (the “empty” stimulus, in which the signal is absent). For instance,  $a$  may be a tilted line segment, and  $b$  the same line segment but vertical, the tilt (which is the signal to be detected) being too small for all answers to be correct. Clearly, the participant in such an experiment cannot first decide that the stimulus being presented now has already been presented before, and that it has to be judged to be  $a$  because so it was before.

With this distinction in mind, however, the formalism (1)-(2)-(3) can be equally applied to both types of situations. In both cases  $a$  is to be replaced with some observable  $A$ , and  $b$  with some observable  $B$  (after which  $a$  and  $b$  per se can be forgotten). The values of  $A$  and  $B$  are the possible responses one records. In the psychophysical example,  $v(A)$  and  $v(B)$  each can attain one of two values: 1 = “I think the stimulus was tilted” or 0 = “I think the stimulus was vertical”. The psychophysical analysis consists in identifying the hit-rate and false-alarm-rate functions (conditioned on the previous stimuli and responses)

$$\begin{aligned} \Pr[v(A) = 1 \text{ in trial } t \mid \text{measurements in trials } 1, \dots, t-1] &= F(\psi^{(t)}, A, 1), \\ \Pr[v(B) = 1 \text{ in trial } t \mid \text{measurements in trials } 1, \dots, t-1] &= F(\psi^{(t)}, B, 1). \end{aligned} \tag{4}$$

The learning (or sequential-effect) aspect of such analysis consists in identifying the function

$$\psi^{(t+1)} = G\left(\psi^{(t)}, S, v\right), \quad S \in \{A, B\}, v \in \{0, 1\}, \quad (5)$$

combined with the “pure” inter-trial dynamics (3).

In the opinion-polling example (say, about Clinton’s and Gore’s honesty), there are two hypothetical observables:  $A$ , corresponding to the question  $a$  = “Is Bill Clinton honest?”, and  $B$ , corresponding to the question  $b$  = “Is Al Gore honest?”, each observable having two possible values, 0 = “Yes” and 1 = “No”. The analysis, formally, is precisely the same as above, except that one no longer uses the terms “hits” and “false alarms” (because “honesty” is not a signal objectively present in one of the two politicians and absent in another).

It is worth noting that in the opinion polling the observables  $A, B$  are defined by the questions  $a, b$  alone only because the allowable responses (Yes or No) and the instructions (“Respond to this question”) do not vary from one trial to another. If the allowable responses varied (e.g., if they were Yes and No in some trials, and Yes, No, and Not Sure in other trials), or if the instruction varied (say, in some trials “Respond as quickly as possible”, in other trials “Think carefully and respond”), they would have also contributed to the identification of the observables. Analogously, in our psychophysics example, the observables are defined by stimuli alone because the instruction to the participants (“Tell us whether the line you see is tilted or vertical”) and the responses allowed (“Tilted” and “Vertical”) remain fixed throughout the successive trials.

In quantum physics, a classical example falling within the same formal scheme as the examples above is one involving measuring the spin of a particle in a given direction. Let the experimenter choose one of two possible directions,  $a$  or  $b$  (unit vectors in space along which the experimenter sets a spin detector). If the particle is a spin-1/2 one, such as an electron, then the spin for each direction chosen can have one of two possible values, 1 = “up” or 0 = “down” (we need not discuss the physical meaning of these designations). These 1 and 0 are then the possible values of the observables  $A$  and  $B$  one associates with the two directions, and the analysis again consists in identifying the functions  $F$ ,  $G$ , and  $H$ .

## 2 A brief account of conventional QM

In QM, all entities operate in a *Hilbert space*, a vector space endowed with the operation of scalar product. The components of the vectors are *complex numbers*. We will assume that the Hilbert spaces to be considered are  $n$ -dimensional ( $n \geq 2$ ), but the generalization of all our considerations to infinite-dimensional spaces is trivial. The scalar product of vectors  $\psi, \phi$  is denoted

$$\langle \psi, \phi \rangle = \sum_{i=1}^n x_i y_i^*,$$

where  $x_i$  and  $y_i$  are components of  $\psi$  and  $\phi$ , respectively, and the star indicates *complex conjugation*: if  $c = a + ib$ , then  $c^* = a - ib$ . The length of a vector  $\phi$  is defined as

$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle}.$$

Any observable  $A$  in this  $n$ -dimensional version of QM is represented by an  $n \times n$  *Hermitian matrix*.<sup>2</sup> This is a matrix with complex entries such that, for any  $i, j \in \{1, \dots, n\}$ ,  $a_{ij} = a_{ji}^*$ . In particular, all diagonal entries of  $A$  are real numbers. For  $n = 2$ , an observable has the form

$$A = \begin{pmatrix} r & x - iy \\ x + iy & s \end{pmatrix},$$

where  $r, s, x, y$  are real numbers.

It is known from matrix algebra that any Hermitian matrix can be uniquely decomposed as

$$A = \sum_{i=1}^k v_i P_i, \quad k \leq n, \quad (6)$$

where  $v_1, \dots, v_k$  are pairwise distinct *eigenvalues* of  $A$  (all real numbers), and  $P_i$  are *eigenprojectors* ( $n \times n$  Hermitian matrices whose eigenvalues are zeros and ones). For any distinct  $i, j \in \{1, \dots, k\}$ , the eigenprojectors satisfy the conditions

$$P_i^2 = P_i \text{ (idempotency), } P_i P_j = \mathbf{0} \text{ (orthogonality)}. \quad (7)$$

Moreover, all eigenprojectors are *positive semidefinite*: for any nonzero vector  $x$ ,  $\langle P_i x, x \rangle \geq 0$ . Finally, all eigenprojectors sum to the identity matrix,

$$\sum_{i=1}^k P_i = I. \quad (8)$$

In QM, the distinct eigenvalues  $v_1, \dots, v_k$  are postulated to form the set of all possible values  $v(A)$ . That is, as a result of measuring  $A$  in any given trial one always observes one of the values  $v_1, \dots, v_k$ . For simplicity (and because all our examples involve binary outcomes), in this paper we will only deal with the observables  $A$  that have two possible values  $v(A)$ , denoted 0 and 1. This means that all our observables can be presented as

$$A = P_1, \quad (9)$$

and

$$P_0^2 = P_0, \quad P_1^2 = P_1, \quad P_0 P_1 = \mathbf{0}, \quad P_0 + P_1 = I. \quad (10)$$

Each eigenvalue  $v$  (0 or 1) has its *multiplicity*  $1 \leq d < n$ . This is the dimensionality of the *eigenspace*  $V$  associated with  $v$ , which is the space spanning the  $d$  pairwise orthogonal *eigenvectors* associated with  $v$  (i.e., the space of all linear combinations of these eigenvectors). Multiplication of  $P_v$  by any vector  $x$  is the orthogonal projection of this vector into  $V$ . If  $d = 1$ , the eigenspace  $V$  is the ray containing a unique unit-length eigenvector of  $A$  corresponding to  $v$ . The eigenvalue  $1 - v$  has the multiplicity  $n - d$ , the dimensionality of the eigenspace  $V^\perp$  which is orthogonal to  $V$ . If both  $d = 1$  and  $n - d = 1$  (i.e.,  $n = 2$ ), then  $A$  is said to have a *non-degenerate spectrum*. In this paper we assume the spectra are generally *degenerate* ( $n \geq 2$ ).

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<sup>2</sup>Each matrix represents an *operator*, in the sense that it transforms a vector by which it is multiplied into another vector. For this reason, the terms “matrix” and “operator” are treated as synonymous (in a finite-dimensional Hilbert space).

The eigenvalues 0, 1 of  $A$  in a given trial generally cannot be predicted, but one can predict the probabilities of their occurrence. To compute these probabilities, QM uses the notion of a *state* of the system. In any given trial the state is unique, and it is represented by a unit length *state vector*  $\psi$ .<sup>3</sup> If the system is in a state  $\psi^{(t)}$  in trial  $t$ , and the measurement is performed on the observable  $A$ , the probabilities of the outcomes of this measurement are given by

$$F(\psi^{(t)}, A, v) = \Pr[v(A) = v \text{ in trial } t \mid \text{measurements in trials } 1, \dots, t-1] = \langle P_v \psi^{(t)}, \psi^{(t)} \rangle = \|P_v \psi^{(t)}\|^2, \quad (11)$$

where  $v = 0, 1$ . Note that these probabilities are conditioned on the previous observables, in trials  $1, \dots, t-1$ , and their observed values.

Given that the observed outcome in trial  $t$  is  $v$ , the state  $\psi^{(t)}$  changes into  $\psi^{(t+1)}$  according to

$$G(\psi^{(t)}, A, v) = \frac{P_v \psi^{(t)}}{\|P_v \psi^{(t)}\|} = \psi^{(t+1)}. \quad (12)$$

This equation represents the von Neumann-Lüders *projection postulate* of QM. The denominator is nonzero because it is the square root of  $\Pr[v(A) = v \text{ in trial } t]$ , and (12) is predicated on  $v$  having been observed. The geometric meaning of  $G(\psi^{(t)}, A, v)$  is that  $\psi^{(t)}$  is orthogonally projected by  $P_v$  into the eigenspace  $V$  and then normalized to unit length.

Finally, the inter-trial dynamics of the state vector in QM (between  $v$  and the next observable, separated by interval  $\Delta$ ) is represented by the *unitary evolution* formula

$$H(\psi^{(t+1)}, \Delta) = U_\Delta \psi^{(t+1)} = \psi_\Delta^{(t+1)}, \quad (13)$$

where  $U_\Delta$  is a *unitary matrix*, defined by the property

$$U_\Delta^{-1} = U_\Delta^\dagger. \quad (14)$$

Here,  $U_\Delta^{-1}$  is the *matrix inverse* ( $U_\Delta^{-1} U_\Delta = U_\Delta U_\Delta^{-1} = I$ ), and  $U_\Delta^\dagger$  is the *conjugate transpose* of  $U_\Delta$ , obtained by transposing  $U_\Delta$  and replacing each entry  $x + iy$  in it with its complex conjugate  $x - iy$ . The unitary matrix  $U_\Delta$  should also be made a function of inter-trial variations in the environment (such as variations in overall noise level, or other participants' responses) if they are non-negligible. The identity matrix  $I$  is a unitary matrix: if  $U_\Delta = I$ , (13) describes *no inter-trial dynamics*, with the state remaining the same through the interval  $\Delta$ .

Note that the eigenvalue  $v$  itself does not enter the computations. This justifies treating it as merely a label for the eigenprojectors and eigenspaces (so instead of 0, 1 we could use any other labels).

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<sup>3</sup>For simplicity, we assume throughout the paper that the system is always in a *pure state*. A more general *mixed state* is represented by a *density matrix*, which is essentially a set of up to  $n$  distinct pure states occurring with some probabilities. The same as with the assumption that  $n$  is finite, the restriction of our analysis to pure states is not critical.

### 3 Measurement sequences, evolution (in)effectiveness, and stability

In this section we introduce terminology and preliminary considerations needed in the subsequent analysis. Throughout the paper we will make use of the following way of describing measurements performed in successive trials:

$$(A_1, v_1, p_1) \rightarrow \dots \rightarrow (A_r, v_r, p_r). \quad (15)$$

We call this a *measurement sequence*. Each triple in the sequence consists of an observable  $A$  being measured, an outcome  $v$  recorded (0 or 1), and its conditional probability  $p$ . The probability is conditioned on the observables measured and the outcomes recorded in the previous trials of the same measurement sequence. Thus,

$$\begin{aligned} p_1 &= \Pr[v(A_1) = v_1 \text{ in trial 1}], \\ p_2 &= \Pr[v(A_2) = v_2 \text{ in trial 2} \mid v(A_1) = v_1 \text{ in trial 1}], \\ p_3 &= \Pr[v(A_3) = v_3 \text{ in trial 3} \mid v(A_1) = v_1 \text{ in trial 1, and } v(A_2) = v_2 \text{ in trial 2}], \\ &\dots \end{aligned} \quad (16)$$

As we assume that the outcomes  $v_1, v_2, \dots$  in a measurement sequence have been recorded, all probabilities  $p_1, p_2, \dots$  are positive if the measurement sequence exists. Recall that the observables  $A_1, A_2, \dots$  in a sequence are uniquely determined by the measurement procedures applied,  $a_1, a_2, \dots$ , and that the outcomes (0 or 1) are eigenvalues of these observables.

Consider now the two-trial measurement sequence

$$(A, v, p) \rightarrow (B, w, q), \quad (17)$$

where  $v, w \in \{0, 1\}$ . Let  $A$  have the eigenprojector matrices  $P_0, P_1$ , and  $B$  have the eigenprojector matrices  $Q_0, Q_1$ . If the initial state of the system is  $\psi = \psi^{(1)}$ , we have

$$p = \|P_v \psi\|^2, \quad (18)$$

and  $\psi^{(1)}$  transforms into

$$\frac{P_v \psi}{\|P_v \psi\|} = \psi^{(2)}. \quad (19)$$

Assuming an interval  $\Delta$  between the two trials,  $\psi^{(2)}$  evolves into

$$\psi_\Delta^{(2)} = U_\Delta \psi^{(2)} = \frac{U_\Delta P_v \psi}{\|P_v \psi\|}. \quad (20)$$

This is the state vector paired with  $B$  in the next measurement, yielding, with the help of some algebra,

$$q = \|Q_w \psi_\Delta^{(2)}\|^2 = \frac{\|Q_w U_\Delta P_v \psi\|^2}{\|P_v \psi\|^2} = \frac{\|(U_\Delta^\dagger Q_w U_\Delta) P_v \psi\|^2}{\|P_v \psi\|^2}. \quad (21)$$

As a special case  $U_\Delta$  can be the identity matrix (no inter-trial changes in the state vector), and then we have

$$q = \frac{\|Q_w P_v \psi\|^2}{\|P_v \psi\|^2}, \quad (22)$$

because in this case

$$(U_\Delta^\dagger Q_w U_\Delta) = Q_w. \quad (23)$$

It is possible, however, that the latter equality holds even if  $U_\Delta^\dagger$  is not the identity matrix. In fact it is easy to see that this happens if and only if  $U_\Delta$  and  $B$  commute, i.e.,

$$U_\Delta B = B U_\Delta. \quad (24)$$

For the proof of this, see Lemma 1 in Appendix.

We will say that

**Definition 1.** A unitary operator  $U_\Delta$  is *ineffective* for an observable  $B$  if the two operators commute.

The justification for this terminology should be transparent: due to (23), in the computation (21) of the probability  $q$  the evolution operator can be ignored, yielding (22). The notion of inefficiency of the evolution operator will play an important role in the analysis of repeated measurements below.

Our next consideration regards the set of all possible values of the initial state vector  $\psi$  for a given measurement sequence. In the applications of QM in physics, this set is assumed to cover the entire Hilbert space in which they are defined. We are not justified to adopt this assumption in psychology, it would be too strong: one could argue that the initial states in a given experiment may be forbidden to attain values within certain areas of the Hilbert space. At the same time, it seems even less reasonable to allow for the possibility that the initial state for a given measurement sequence is always fixed at one particular value. The initial state vectors, as follows from both the QM principles and common sense, should depend on the system's history prior to the given experiment, and this should create some variability from one replication of this experiment to another. This is important, because, given a set of observables, specially chosen initial state vectors may exhibit “atypical” behaviors, those that would disappear if the state vector were modified even slightly.

This leads us to adopting the following

**Stability Principle:** *If  $\psi$  is a possible initial state vector for a given measurement sequence in an  $n$ -dimensional Hilbert space, then there is an open ball  $B_r(\psi)$  centered at  $\psi$  with a sufficiently small radius  $r$ , such that any vector  $\psi + \delta$  in this ball, normalized by its length  $\|\psi + \delta\|$ , is also a possible initial state vector for this measurement sequence.*

We will say that

**Definition 2.** A property of a measurement sequence is (or holds) *stable* for an initial vector  $\psi$ , if it holds for all state vectors within a sufficiently small  $B_r(\psi)$ .



Almost all our propositions below are proved under this stability clause, specifically by using the reasoning presented in Lemma 2 in Appendix.

## 4 Consequences for “ $\mathbf{a} \rightarrow \mathbf{a}$ ”-type measurement sequences

Using the definitions and the language just introduced, we will now focus on the consequences of (11)-(12)-(13) for repeated measurements with repeated responses,

$$(A, v, p) \rightarrow (A, v, p'). \quad (25)$$

Consider an opinion-polling experiment, with questions like  $a$  = “Is Bill Clinton trustworthy?” [27]. As argued for in Introduction, if the same question is posed twice,  $a \rightarrow a$ , a typical respondent, who perhaps hesitated when choosing the response the first time she was asked  $a$ , would now certainly be expected to repeat it, perhaps with some display of surprise at being asked the question she has just answered. This may not be true for all possible questions, but it is certainly true for a vast class thereof. Let us formulate this as

**Property 1** *For some nonempty class of questions, if a question is repeated twice in successive trials (separated by one of a broad range of inter-trial intervals), the response to it will also be repeated.*

If a question  $a$  within the scope of this property is represented by an observable  $A$ , we are dealing with the measurement sequence (25) in which  $p' = 1$ . Such a measurement sequence does not disagree with the formulas (11)-(12)-(13), and in fact is even predicted by them if the intervening inter-trial evolution of the state vector is assumed to be ineffective. Indeed, (21) for the measurement sequence (25) acquires the form

$$p' = \frac{\| (U_\Delta^\dagger P_v U_\Delta) P_v \psi \|^2}{\| P_v \psi \|^2}, \quad (26)$$

and the inefficiency of  $U_\Delta$  for  $A$  implies

$$p' = \frac{\| P_v^2 \psi \|^2}{\| P_v \psi \|^2} = 1, \quad (27)$$

because  $P_v^2 = P_v$  holds for all projection operators.

This is easy to understand informally. An outcome  $v$  observed in the first measurement,  $(A, v, p)$ , is associated with an eigenspace  $V$ . The measurement orthogonally projects the state vector  $\psi = \psi^{(1)}$  into this eigenspace, and this projection is normalized to become the new state  $\psi^{(2)}$ . The application of the same measurement to  $\psi^{(2)}$  orthogonally projects it into  $V$  again, but since  $\psi^{(2)}$  is already in  $V$ , it does not change. The squared length of the projection therefore is 1, and this is what the probability  $p'$  is.

As it turns out, under the stability principle, effective inter-trial evolution is in fact excluded for the observables representing the questions falling within the scope of Property 1. In other words, for all such questions, the unitary operators  $U_\Delta$  can be ignored in all probability computations.

Let us say that

**Definition 3.** An observable  $A$  has the *Lüders property* with respect to a state vector  $\psi$  if the existence of the measurement  $(A, v, p)$  for this  $\psi$  and an outcome  $v \in \{0, 1\}$  implies that the property  $p' = 1$  holds stable for this  $\psi$  in the measurement sequence  $(A, v, p) \rightarrow (A, v, p')$ .

In other words, the Lüders property means that an answer to a question (represented by  $A$ ) is repeated if the question is repeated, and that this is true not just for one initial state vector  $\psi$ , but for all state vectors sufficiently close to it.

We now can formulate our first proposition.

**Proposition 1** (repeated measurements). *An observable  $A$  has the Lüders property if and only if  $U_\Delta$  in (13) is ineffective for  $A$ .*

See Appendix for a formal proof. In the formulation of Property 1, the interval  $\Delta$  and the question represented by  $A$  can vary within some broad limits, whence the inefficiency of  $U_\Delta$  for  $A$  should also hold for each of these intervals combined with each of these questions.

We have to be careful not to overgeneralize the Lüders property and the ensuing inefficiency property. As we discussed in Introduction, one can think of situations where replications of a question may lead the respondent to “change her mind.” The most striking contrast, however, is provided by psychophysical applications of QM. Here, the inter-trial dynamics not only cannot be ignored, it must play a central role.

Let us illustrate this on an old but very thorough study by Atkinson, Carterette, and Kinchla [28]. In the experiments they report, each stimulus consisted of two side-by-side identical fields of luminance  $L$ , to one of which a small luminance increment  $\Delta L$  could be added, serving as the signal to be detected. There were three stimuli:

$$a = (L + \Delta L, L), \quad b = (L, L + \Delta L), \quad c = (L, L). \quad (28)$$

In each trial the observer indicated which of the two fields, right one or left one, contained the signal. There were thus two possible responses: Left and Right. An application of QM analysis to these experiments requires  $a, b, c$  to be translated into observable  $A, B, C$ , each with two eigenvalues, say,  $0 = \text{Left}$  and  $1 = \text{Right}$ . In the experiments we consider no feedback was given to the observers following a response. This is a desirable feature. It makes the sequence of trials we consider formally comparable to successive measurements of spins in quantum physics: measurements simply follow each other, with no interventions in between.<sup>4</sup>

We are interested in measurement sequences

$$\begin{aligned} (A, 0, p_1) &\rightarrow (A, 0, p'_1), & (A, 1, p_2) &\rightarrow (A, 1, p'_2), \\ (B, 0, p_3) &\rightarrow (B, 0, p'_3), & (B, 1, p_4) &\rightarrow (B, 1, p'_4), \\ (C, 0, p_5) &\rightarrow (C, 0, p'_5), & (C, 1, p_6) &\rightarrow (C, 1, p'_6). \end{aligned} \quad (29)$$

Recall that the probabilities  $p'_i$  ( $i = 1, \dots, 6$ ) are conditioned on previous measurements, so that, e.g.,  $p'_1 + p'_2 \neq 1$  while  $p_1 + p_2 = 1$ .

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<sup>4</sup>However, this precaution seems unnecessary, as the results of the experiments with feedback in Ref. [28] do not qualitatively differ from the ones we discuss here.

For each observer, the probabilities were estimated from the last 400 trials out of 800 (to ensure an “asymptotic” level of performance). The results, averaged over 24 observers, were as follows:

Experiment 1			Experiment 2		
index	$p$	$p'$	index	$p$	$p'$
1	.65	.73	1	.56	.70
2	.35	.38	2	.44	.41
3	.36	.39	3	.27	.31
4	.64	.71	4	.73	.79
5	.50	.53	5	.39	.50
6	.50	.60	6	.61	.65

The leftmost column in each table corresponds to the index associated with  $p$  and  $p'$  in (29). Thus, the first row shows  $p_1$  and  $p'_1$ , the last one shows  $p_6$  and  $p'_6$ . The two experiments differed in one respect only: in Experiment 1 the stimuli  $a$  and  $b$  were presented with equal probabilities, while in Experiment 2 the stimulus  $b$  was three times more probable than  $a$  (the probability of  $c$  was 0.2 in both experiments). The results show, in accordance with conventional detection models, that this manipulation makes responses in Experiment 2 biased towards the correct response to  $b$ . This aspect of the data, however, is not of any significance for us. What is significant, is that, in accordance with Proposition 1, we should conclude that the inter-trial evolution (13) here intervenes always and significantly.

## 5 A consequence for “ $a \rightarrow b \rightarrow a$ ”-type measurement sequences

Returning to the opinion polling experiments, consider the situation involving two questions, such as  $a$  = “Is Bill Clinton honest?” and  $b$  = “Is Al Gore honest?” The two questions are posed in one of the two orders,  $a \rightarrow b$  or  $b \rightarrow a$ , to a large group of people. The same as with asking the same question twice in a row, one would normally consider it unnecessary to extend these sequences by asking one of the two questions again, by repeating  $b$  or  $a$  after having asked  $a$  and  $b$ . A typical respondent, again, will be expected to repeat her first response. We find it “almost certain” (the “almost” being inserted here because we cannot refer to any systematic experimental study of this obvious expectation) that from the nonempty (in reality, vast) class of questions falling within the scope of Property 1 one can always choose pairs of questions falling within the scope of the following extension of this property.

**Property 2.** *Within a nonempty subclass of questions (and for the same set of inter-trial intervals) for which Property 1 holds, if a question  $a$  is asked following questions  $a$  and  $b$  (in either order), the response to it will necessarily be the same as that given to the question  $a$  the first time it was asked.*

As always, we replace  $a, b$  with observables  $A, B$ , and use the following notation: the probability of obtaining a value  $v$  when measuring the observable  $A$  is denoted  $p_{vA}$ ,  $q_{vA}$ , etc. (the letters  $p, q$ , etc. distinguishing different measurements);

we use analogous notation for the probability of obtaining a value  $w$  when measuring the observable  $B$ .

Consider the measurement sequence

$$(A, v, p_{vA}) \rightarrow (B, w, p_{wB}) \rightarrow (A, v, p'_{vA}) \quad (30)$$

Property 2 implies that in these sequences  $p'_{vA} = 1$  and  $q'_{wA} = 1$ . As it turns out, this property has an important consequence (assuming the two inter-trial intervals in the measurement sequences belong to the same class as  $\Delta$  in Proposition 1).

**Proposition 2** (alternating measurements). *Let  $A$  and  $B$  possess the Lüders property, and let the measurement sequences*

$$(A, v, p_{vA}) \rightarrow (B, w, p_{wB}) \quad (31)$$

*exist for all  $v, w \in \{0, 1\}$ , and some initial state vector  $\psi$ . Then, in the measurement sequences (30), the property  $p'_{vA} = 1$  holds stable for this  $\psi$  if and only if  $A$  and  $B$  commute,  $AB = BA$ .*

In other words, if the probabilities  $p_{vA}, p_{wB}, q_{wB}, q_{vA}$  are nonzero in (31) for some  $\psi$ , the sequences (30) exist with  $p'_{vA} = 1$  and  $q'_{wA} = 1$  for all state vectors in a small neighborhood of  $\psi$  if and only if  $AB = BA$ . See Appendix for a formal proof.

To illustrate how this works, recall that  $A$  and  $B$  commute if and only if they have one and the same set of orthonormal eigenvectors  $e_1, \dots, e_n$  (generally, not unique). Since  $A$  and  $B$  have two eigenvalues each, the difference between the two observables is in how these eigenvectors are grouped into two eigenspaces. Take one of the measurement sequences of the (30)-type, say,

$$(A, 1, p_{1a}) \rightarrow (B, 0, p_{0B}) \rightarrow (A, 1, p'_{1a}). \quad (32)$$

Since  $A$  and  $B$  have the Lüders property, all the probabilities are the same as if there was no inter-trial dynamics involved. Proceeding under this assumption, the first measurement projects the initial vector  $\psi = \psi^{(1)}$  into  $V_1$  that spans some of the vectors  $e_1, \dots, e_n$ . Let this projection (after its length was normalized to 1) be  $\psi^{(2)}$ . The second measurement projects  $\psi^{(2)}$  into the intersection  $V_1 \cap W_0$  that spans a smaller subset of these vectors. The third measurement then, since the second normalized projection  $\psi^{(3)}$  is already in  $V_1$ , does not change it,  $\psi^{(4)} = \psi^{(3)}$ . This means that the third probability,  $p'_{1a}$ , being the scalar product of  $\psi^{(3)}$  and  $\psi^{(4)}$ , must be unity.

The commutativity of  $A$  and  $B$  is important because it has an experimentally testable consequence.

**Proposition 3** (no order effect). *If  $A$  and  $B$  possessing the Lüders property commute, then in the measurement sequences*

$$(A, v, p_{vA}) \rightarrow (B, w, p_{wB}),$$

$$(B, w, q_{wB}) \rightarrow (A, v, q_{vA})$$

the joint probabilities of the two outcomes are the same,

$$p_{vA}p_{wB} = q_{wB}q_{vA}. \quad (33)$$

Consequently,

$$\Pr[v(A) = v \text{ in trial 1}] = \Pr[v(A) = v \text{ in trial 2}] \quad (34)$$

and

$$\Pr[w(B) = w \text{ in trial 1}] = \Pr[w(B) = w \text{ in trial 2}]. \quad (35)$$

To clearly understand what is being stated, recall that  $p_{wB}$  is the conditional probability of observing the value  $w$  of  $B$  given that before this the outcome was the value  $v$  of  $A$ . So, the product of  $p_{vA}p_{wB}$  is the overall probability of the first of the two sequences in the proposition. The value of  $q_{wB}q_{vA}$  is understood analogously. Equation (33) therefore states that

$$\Pr[v(A) = v \text{ in trial 1 and } w(B) = w \text{ in trial 2}] = \Pr[w(B) = w \text{ in trial 1 and } v(A) = v \text{ in trial 2}].$$

The proof of the proposition is given in Appendix.

Equations (33)-(34)-(35) are empirically testable predictions. Moreover, if we assume that the questions like “Is Clinton honest” and “Is Gore honest” fall within the scope of Property 2 (and it would be amazing if they did not), these predictions are known to be de facto falsified.

**Property 3.** *Within a nonempty subclass of questions for which Property 2 holds (and for the same set of inter-trial intervals), the joint probability of two successive responses depends on the order in which the questions were posed.*

This “question order effect” has in fact been presented as one for whose understanding QM is especially useful: the empirical finding that  $p_{vA}p_{wB} \neq q_{wB}q_{vA}$  is explained in Ref. [24] by assuming that  $A$  and  $B$  do not commute. In the survey reported by Moore [27], about 1,000 people were asked two questions, one half of them in one order, the other half in another. The probability estimates are presented for four pairs of questions: the first pair was about the honesty of Bill Clinton ( $a$ ) and Al Gore ( $b$ ), the second about the honesty of Newt Gingrich ( $a$ ) and Bob Dole ( $b$ ), etc.

question pair	probability of Yes to $a$		question pair	probability of Yes to $b$	
	in $a \rightarrow b$	in $b \rightarrow a$		in $a \rightarrow b$	in $b \rightarrow a$
1	.50	.57	1	.60	.68
2	.41	.33	2	.64	.60
3	.41	.53	3	.56	.46
4	.64	.52	4	.33	.45

The results are presented in the form of  $\Pr[v(A) = 1 \text{ in trial } i]$  and  $\Pr[w(B) = 1 \text{ in trial } i]$ ,  $i = 1, 2$ , so the tested predictions are (34) and (35). As we can see, for all question pairs, the probability estimates of Yes to the same question

differ depending on whether the question was asked first or second. Given the sample size (about 500 respondents per question pair in a given order) the differences are not attributable to chance variation.

Properties 1, 2, and 3 turn out to be incompatible within the framework of QM. We should conclude therefore that QM cannot be applied to the questions that have these properties without modifications.

## 6 Would POVMs work?

Are there more flexible versions (generalizations) of QM that could be used instead?

One widely used generalization of the conventional QM involves replacing the projection operators with *positive-operator-valued measures* (POVMs), see, e.g., Refs. [29] and [30]. The conceptual set-up here is as follows. We continue to deal with an  $n$ -dimensional Hilbert space ( $n \geq 2$ ). The notion of a state represented by a unit vector  $\psi$  in this space remains unchanged. The generalization occurs in the notion of an observable. For experiments with binary outcomes, an observable  $A$  of the conventional QM is defined by (9), with eigenvalues  $(0, 1)$  and eigenprojectors  $(P_0, P_1)$ . The eigenvalues themselves are not relevant insofar as they are distinct: replacing  $0, 1$  with another pair of distinct values amounts to trivial relabeling of the measurement outcomes. The information about the observable  $A$  therefore is contained in the eigenprojectors  $P_0, P_1$ . They are Hermitian positive semidefinite operators subject to the restrictions (10).

A generalized observable, or POVM,  $A$  (continuing to consider only binary outcomes) is defined as a pair  $(E_0, E_1)$  of Hermitian positive semidefinite operators in the  $n$ -dimensional Hilbert space, summing to the identity matrix  $I$ . In other words, the generalization from eigenprojectors  $P_v$  to POVM components  $E_v$  amounts to dropping the idempotency and orthogonality constraints, defined in (7).<sup>5</sup>

Any component  $E_v$  ( $v = 0, 1$ ) can be presented as  $M_v^\dagger M_v$ , where  $M_v$  is some matrix and  $M_v^\dagger$  is its conjugate transpose. The representation  $E = M_v^\dagger M_v$  for a given  $E_v$  is not unique, but it is supposed to be fixed within a given experiment (i.e., for a given measurement procedure).

The measurement formulas specifying  $F$  and  $G$  in (1)-(2) can now be formulated to resemble (11)-(12). The conditional probability of an outcome  $v = 0, 1$  of the measurement of  $A = (E_0, E_1)$  in state  $\psi^{(t)}$  is

$$F(\psi^{(t)}, A, v) = \Pr[v(A) = v \text{ in trial } t \mid \text{measurements in trials } 1, \dots, t-1] = \langle E_v \psi^{(t)}, \psi^{(t)} \rangle = \|M_v \psi^{(t)}\|^2 \quad (36)$$

This measurement transforms  $\psi^{(t)}$  into

$$G(\psi^{(t)}, A, v) = \frac{M_v \psi^{(t)}}{\|M_v \psi^{(t)}\|} = \psi^{(t+1)}. \quad (37)$$

The formula for the evolution of the state vector between trials remains the same as for the conventional observables, (13).

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<sup>5</sup>Without getting into details, the theory of POVMs is sometimes referred to as the *open-system* QM because of *Naimark's dilation theorem* [31]. It says that any POVM  $A$  in a Hilbert space  $H$  can be represented by a conventional quantum observable  $\tilde{A}$  in a Hilbert space of higher dimensionality, the tensor product  $H \otimes K$  of the original Hilbert space  $H$  and another Hilbert space  $K$ . The latter is interpreted as an *environment* for  $H$ . The measurements of  $\tilde{A}$  are performed in a conventional way in  $H \otimes K$ , and the resulting state vectors are projected back into  $H$ .

It is easy to see that we no longer need to involve inter-trial changes in the state vector to explain the fact that, in psychophysics, a replication of stimulus does not lead to the replication of response. In a measurement sequence

$$(A, v, p) \rightarrow (A, v, p'),$$

if  $U_\Delta$  is the identity matrix, then  $p'$  is given by

$$p' = \left\langle E_v \frac{M_v \psi}{\|M_v \psi\|}, \frac{M_v \psi}{\|M_v \psi\|} \right\rangle = \frac{\langle (M_v^\dagger)^2 M_v^2 \psi, \psi \rangle}{\langle M_v^\dagger M_v \psi, \psi \rangle}.$$

This value is generally different from 1: since  $(M_v^\dagger)^2 M_v^2$ , where  $M_v$  is not necessarily an orthogonal projector, is generally different from  $M_v^\dagger M_v$ ,  $\langle (M_v^\dagger)^2 M_v^2 \psi, \psi \rangle$  is generally different from  $\langle M_v^\dagger M_v \psi, \psi \rangle$ .

This is interesting, as it suggests the possibility of treating psychophysical judgments and opinion polling within the same (evolution-free) framework. This encouraging possibility, however, cannot be realized: the theory of POVMs cannot help us in reconciling Properties 2 and 3 in opinion-polling, because POVMs with Lüders property cannot be anything but conventional observables. This is shown in the following

**Proposition 4** (no generalization). *A POVM  $A = (E_0, E_1)$  has the Lüders property with respect to a state  $\psi$  if and only if  $A$  is a conventional observable (i.e., it is a Hermitian operator, and its components  $E_0, E_1$  are its eigenprojectors).*

See Appendix for a proof.

Proposition 4 says that POVMs to be used to model opinion polling should be conventional observables, otherwise Property 1 will be necessarily contradicted. But then Propositions 1 and 2 are applicable, and they say that the inter-trial dynamics is ineffective, and that all the observables representing different questions within the scope of Property 2 pairwise commute. This, in turn, allows us to invoke Proposition 3, with the result that, contrary to Property 3, the order of the questions should have no effect on the response probabilities.

## 7 Conclusion

Let us summarize. Both cognitive science and quantum physics deal with fundamentally probabilistic input-output relations, exhibiting a variety of sequential effects. Both deal with these relations and effects by using, in some form or another, the notion of an “internal state” of a system. In psychology, the maximally general version is provided by the probabilistic generalization of the old behaviorist S-O-R scheme: the probability of an output is a function of the input and the system’s current state (function  $F$  in (1)), and both the input and the output change the current state into a new state (function  $G$  in (2)). If we discretize behavior into subsequent trials, then we need also a function describing how the state of the system changes between the trials (function  $H$  in (3)).

Quantum physics uses a special form of the functions  $F$ ,  $G$ , and  $H$ , the ones derived from (or constituting, depending on the axiomatization) the principles of QM. Functions  $F$  and  $G$  are given by (11)-(12) in the conventional QM, and by (36)-(37) in the QM with POVMs, with the inter-trial evolution in both cases described by (13). Nothing a priori

precludes these special forms of  $F, G, H$  from being applicable in cognitive science, and such applications were successfully tried: by appropriately choosing observables and states, certain experimental data in human decision making were found to conform with QM predictions [22].

As this paper shows, however, QM encounters difficulties in accounting for some very basic empirical properties. In opinion polling (more generally, in all psychological tasks where stimuli/questions can be confidently identified by features other than those being judged), there is a class of questions such that a repeated question is answered in the same way as the first time it was asked. This agrees with the Lüders projection postulate, and renders the use of both the inter-trial dynamics of the state vector and the POVM theory unnecessary: to have this property the questions asked have to be represented by conventional observables with ineffective inter-trial dynamics. In many situations, we also expect that for a certain class of questions the response to two replications of a given question remains the same even if we insert another question in between and have it answered. This property can only be handled by QM if the conventional observables representing different questions all pairwise commute, i.e., can be assigned the same set of eigenvectors. This, in turn, leads to a strong prediction: the joint probability of two responses to two successive questions does not depend on their order. This prediction is known to be violated for some pairs of questions. The explanation of the “question order effect” is in fact one of the most successful applications of QM in psychology [24], but it requires noncommuting observables, and these, as we have seen, cannot account for the repeated answers to repeated questions.<sup>6</sup>

Our paper in no way dismisses the applications of QM in cognitive psychology, or diminishes their modeling value. It merely sounds a cautionary note: it seems that we lack a deeper theoretical foundation, a set of well-justified principles that would determine where QM can and where it must not be used. We should also point out that the problems identified in this paper are not unique to QM. For example, random utility theories also have difficulty explaining the trial to trial dependencies in answers to questions. If we assume, as done in traditional random utility theories (see, e.g., Ref. [32]), that a response is based on a randomly sampled utility in each trial, then repeating the response will produce different random samples in each trial. That is why in the experiments designed to test random utility models questions never repeated back to back, and instead “filler trials” are inserted to make participants forget their earlier choice.

Clearly, the basic properties that we have shown to contravene QM can be “explained away” by invoking considerations formulated in traditional psychological terms. One can, e.g., dismiss the problem with repeated questions in opinion polling by pointing out that the respondents “merely” remember their previous answers and “simply” do not want to contradict themselves. One can similarly dismiss the question order effect by pointing out that the first question “simply” changes the context for pondering the second question, e.g., reminds something the respondent would not have thought of had the second question been asked first. These may very well be valid considerations. But if one allows for such extraneous to QM explanations, one needs to understand (A) why the same extraneous considerations do not intervene in situations where QM is successfully applicable, and (B) why one cannot stick to considerations of this kind and dispense with QM altogether.

A reasonable answer is that the value of QM applications is precisely in that it replaces the disparate conventional psychological notions with unified and mathematically rigorous ones. But then in those situations where we find QM

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<sup>6</sup>Some of these issues were previously raised in a more informal manner by Geoff Iverson (personal communication, June 2010).



not applicable one needs more than invoking these conventional psychological notions. One needs principles. Both in a psychophysical detection experiment and in opinion polling, participants may think of various things between trials, and previously presented stimuli/questions as well as previously given responses definitely change something in their mind, affecting their responses to subsequent stimuli/questions. Why then the applicability of QM is not the same in these two cases? Why, e.g., should the inter-trials dynamics of the state vector (or the use of POVMs in place of conventional observables) be critical in one case and ineffective (or unnecessary) in another?

One should also consider the possibility that rather than acting as switches distinguishing the situations in which (11)-(12) or (36)-(37) are and are not applicable, the set of the hypothetical principles in question may require a higher level of generality for the functions  $F, G, H$ . A serious and meticulous work is needed therefore to determine precisely what features of QM are critical for this or that (un)successful explanation. As an example, virtually any functions  $F, G, H$  in the general formulas (1)-(2)-(3) predict the existence of the question order effect, and the functions can always be adjusted to account for any specific effect. The QQ constraint for the question order effect discovered by Wang and Busemeyer [24] means that, for any two questions  $a, b$  and any respective responses  $v, w \in \{0, 1\}$ ,

$$h_{ab}(v, w) = \Pr[v \text{ in response to } a \text{ in trial 1, and } w \text{ in response to } b \text{ in trial 2}] = f_{ab}(v, w) + g_{ab}(v, w),$$

where

$$f_{ab}(v, w) = f_{ba}(w, v), \quad (38)$$

and

$$g_{ab}(v, w) = -g_{ab}(1 - v, 1 - w). \quad (39)$$

It follows then that

$$h_{ab}(v, w) + h_{ab}(1 - v, 1 - w) = h_{ba}(w, v) + h_{ba}(1 - w, 1 - v),$$

which is the QQ equation. Clearly,  $F, G, H$  functions in (1)-(2)-(3) can be chosen so that  $f_{ab}$  and  $g_{ab}$  have the desired symmetry properties, and the QM version of  $F$  and  $G$  used in Ref. [24] (with ineffective  $H$ ) is only one way of achieving this. It is an open question whether one of many possible generalizations of this QM version may turn out more profitable for dealing with opinion polling.

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## Appendix: Proofs

**Proposition 1** (repeated measurements) *An observable  $A$  has the Lüders property if and only if  $U_\Delta$  in (13) is ineffective for  $A$ .*

*Proof.* The “if” part is demonstrated by (26)-(27). For the “only if” part, let the eigenprojector  $P$  correspond to the eigenvalue  $v$  of  $A$ . We have

$$p = \langle P\psi, \psi \rangle = \|P\psi\|^2 > 0,$$

and the next state vector is

$$\psi^{(2)} = \frac{P\psi}{\|P\psi\|}.$$

Following the evolution

$$\psi^{(2)} \rightarrow \psi_\Delta^{(2)} = U_\Delta \psi^{(2)},$$

the Lüders property implies

$$p' = \langle P\psi_\Delta^{(2)}, \psi_\Delta^{(2)} \rangle = 1,$$

or, equivalently,

$$\langle PU_\Delta \psi^{(2)}, U_\Delta \psi^{(2)} \rangle = \langle U_\Delta^\dagger PU_\Delta \psi^{(2)}, \psi^{(2)} \rangle = 1.$$

As  $\|\psi^{(2)}\| = 1$ , and  $U_\Delta^\dagger PU_\Delta$  is an orthogonal projection, the lengths  $\|U_\Delta^\dagger PU_\Delta \psi^{(2)}\|$  does not exceed 1. Therefore  $\langle U_\Delta^\dagger PU_\Delta \psi^{(2)}, \psi^{(2)} \rangle = 1$  implies

$$U_\Delta^\dagger PU_\Delta \psi^{(2)} = \psi^{(2)},$$

or

$$U_\Delta^\dagger PU_\Delta P\psi = P\psi.$$

By the stability considerations (Lemma 2 below, with  $X_1 = P$  to guarantee  $p > 0$ , and  $Y = U_\Delta^\dagger PU_\Delta, Z = P$ ),

$$U_\Delta^\dagger PU_\Delta P = P.$$

Since  $P$  is Hermitian

$$P^\dagger = P \left( U_\Delta^\dagger PU_\Delta \right) = P = \left( U_\Delta^\dagger PU_\Delta \right) P,$$

so  $P$  and  $U_\Delta^\dagger P U_\Delta$  commute. Now,  $U_\Delta^\dagger (I - P) U_\Delta = I - U_\Delta^\dagger P U_\Delta$ , and it commutes with  $I - P$ :

$$(I - U_\Delta^\dagger P U_\Delta)(I - P) = I - U_\Delta^\dagger P U_\Delta - P + (U_\Delta^\dagger P U_\Delta)P = I - U_\Delta^\dagger P U_\Delta - P + P(U_\Delta^\dagger P U_\Delta) = (I - P)(I - U_\Delta^\dagger P U_\Delta).$$

Let us choose an orthonormal basis  $e_1, \dots, e_n$  consisting of the eigenvectors of  $P$ , so that  $e_1, \dots, e_k$  are the eigenvectors associated with eigenvalue 1 (and then the rest of the  $e$ 's are the eigenvectors of  $I - P$  with eigenvalue 1). In this basis,  $P$  is a diagonal matrix with the first  $k$  diagonal entries 1, and the rest of them zero, and  $I - P$  is a diagonal matrix with the last  $n - k$  diagonal entries 1, and the rest of them zero. We have  $P e_i = e_i$ , for  $i \leq k$ , and then

$$(U_\Delta^\dagger P U_\Delta) P e_i = (U_\Delta^\dagger P U_\Delta) e_i = e_i.$$

So, all  $e_1, \dots, e_k$  are eigenvectors of  $U_\Delta^\dagger P U_\Delta$  with eigenvalues 1. Since  $U_\Delta^\dagger P U_\Delta$  has the same eigenvectors  $e_1, \dots, e_n$  as  $P$ , it is a diagonal matrix with the first  $k$  diagonal entries 1. Analogously we find that  $U_\Delta^\dagger (I - P) U_\Delta$  is a diagonal matrix with the last  $n - k$  diagonal entries 1. Since these matrices add to  $I$ , the rest of the diagonal entries in  $U_\Delta^\dagger P U_\Delta$  must be zero, and this means that  $U_\Delta^\dagger P U_\Delta = P$ . By Lemma 1, this means that  $U_\Delta$  and  $A$  commute.  $\square$

**Proposition 2** (alternating measurements). *Let  $A$  and  $B$  possess the Lüders property, and let the measurement sequences*

$$(A, v, p_{vA}) \rightarrow (B, w, p_{wB})$$

*exist for all  $v, w \in \{0, 1\}$ , and some initial state vector  $\psi$ . Then, in the measurement sequences*

$$(A, v, p_{vA}) \rightarrow (B, w, p_{wB}) \rightarrow (A, v, p'_{vA}),$$

*the property  $p'_{vA} = 1$  holds stable for this  $\psi$  if and only if  $A$  and  $B$  commute,  $AB = BA$ .*

*Proof.* Let the eigenprojectors  $P$  and  $Q$  correspond to the eigenvalue 0 of  $A$  and  $B$ , respectively. Consider the measurement sequence

$$(A, 0, p_{0A}) \rightarrow (B, 0, p_{0B}) \rightarrow (A, 0, p'_{0A}).$$

Let the two inter-trial intervals be  $\Delta_1$  and  $\Delta_2$ . Due to the Lüders property, each of the corresponding evolution operators  $U_1$  and  $U_2$  is ineffective for both  $A$  and  $B$  (commutes with any of their eigenprojectors). Then the state vectors at the beginning of each trial are

$$\psi = \psi^{(1)} \rightarrow U_1 \frac{P_v \psi}{\|P_v \psi\|} = \psi_{\Delta_1}^{(2)} \rightarrow U_2 U_1 \frac{Q_w P_v \psi}{\|Q_w U_1 P_v \psi\|} = \psi_{\Delta_2}^{(3)}$$

and the corresponding probabilities are

$$p_{0A} = \|P\psi\|^2, \quad p_{0B} = \frac{\langle QP\psi, P\psi \rangle}{\|P\psi\|^2}, \quad p'_{0A} = \frac{\langle PQP\psi, QP\psi \rangle}{\|QP\psi\|^2}.$$

The “if” part is proved by direct computation. If  $AB = BA$ , then  $PQ = QP$ , and we have

$$p'_{0A} = \frac{\langle PQP\psi, QP\psi \rangle}{\|QP\psi\|^2} = \frac{\langle QP\psi, QP\psi \rangle}{\|QP\psi\|^2} = 1.$$

We now prove the “only if” part. Denoting  $\phi = QP\psi/\|QP\psi\|$ , the condition

$$\langle P\phi, \phi \rangle = p'_{0A} = 1$$

implies  $P\phi = \phi$ , because  $\|\phi\| = 1$ , and  $\|P\phi\|$  (the length of an orthogonal projection of  $\phi$ ) does not exceed 1. Hence

$$PQP\psi = QP\psi.$$

By the stability considerations (Lemma 2 below, with  $X_1 = P, X_2 = PQP$  to guarantee  $p_{0A} > 0, p_{0B} > 0$ , and  $Y = PQP, Z = QP$ ),

$$PQP = QP,$$

and, taking the conjugate transpositions,

$$PQP = (PQP)^\dagger = PQ.$$

So,  $PQ = QP$ . By simple algebra then, either of  $P, I - P$  commutes with either of  $Q, I - Q$ , and this means that  $AB = BA$ .  $\square$

**Proposition 3** (no order effect). *If  $A$  and  $B$  possessing the Lüders property commute, then in the measurement sequences*

$$(A, v, p_{vA}) \rightarrow (B, w, p_{wB}),$$

$$(B, w, q_{wB}) \rightarrow (A, v, q_{vA})$$

*the joint probabilities of the two outcomes are the same,*

$$p_{vA}p_{wB} = q_{wB}q_{vA}.$$

*Consequently,*

$$\Pr[v(A) = v \text{ in trial 1}] = \Pr[v(A) = v \text{ in trial 2}]$$

*and*

$$\Pr[w(B) = w \text{ in trial 1}] = \Pr[w(B) = w \text{ in trial 2}].$$

*Proof.* For the sequence  $(A, v, p_{vA}) \rightarrow (B, w, p_{wB})$ , we have

$$\psi = \psi^{(1)} \rightarrow U_A \frac{P\psi}{\|P\psi\|} = \psi_{\Delta_A}^{(2)}$$

and the probabilities are

$$p_{0A} = \|P\psi\|^2, \\ p_{0B} = \langle Q\psi_{\Delta_A}^{(2)}, \psi_{\Delta_A}^{(2)} \rangle = \langle QU_A \frac{P\psi}{\|P\psi\|}, U_A \frac{P\psi}{\|P\psi\|} \rangle = \frac{\langle QU_AP\psi, U_AP\psi \rangle}{\|P\psi\|^2}.$$

The joint probability is therefore

$$p_{0A}p_{0B} = \langle QU_AP\psi, U_AP\psi \rangle = \langle PQP\psi, \psi \rangle,$$

where we have used the commutativity of the unitary operators with the observables. Analogously, for the sequence  $(B, w, q_{wB}) \rightarrow (A, v, q_{vA})$ , we get

$$p_{0B}p_{0A} = \langle QPQ\psi, \psi \rangle.$$

But  $P$  and  $Q$  commute by Proposition 2, whence

$$PQP = QPQ.$$

This proves  $p_{vA}p_{wB} = q_{wB}q_{vA}$ . The other two equations follow by presenting the probabilities in them as sums of suitably chosen joint probabilities.  $\square$

**Proposition 4** (no generalization) *A POVM  $A = (E_0, E_1)$  has the Lüders property with respect to a state  $\psi$  if and only if  $A$  is a conventional observable (i.e., it is a Hermitian operator, and its components  $E_0, E_1$  are its eigenprojectors).*

*Proof.* The “if” part is obvious: if  $A$  is a Hermitian operator, it has the Lüders property with respect to any state  $\psi$ .

We prove the “only if” part. Consider the measurement sequence

$$(A, v, p_v) \rightarrow (A, v, p'_v),$$

with  $\psi = \psi^{(1)}$ . Since

$$p_v = \langle E_v\psi, \psi \rangle = \langle M_v\psi, M_v\psi \rangle = \|M_v\psi\|^2 > 0,$$

the next state vector (interjecting the unitary evolution operator  $U_\Delta$ ) is

$$\psi_{\Delta}^{(2)} = U_\Delta \psi^{(2)} = U_\Delta \frac{M_v\psi}{\|M_v\psi\|} = \frac{U_\Delta M_v\psi}{\|M_v\psi\|}.$$

Since  $\|\psi_{\Delta}^{(2)}\| = 1$ , it follows from Lemma 3 below that the equality  $p'_v = \langle E_v \psi_{\Delta}^{(2)}, \psi_{\Delta}^{(2)} \rangle = 1$  implies  $E_v \psi_{\Delta}^{(2)} = \psi_{\Delta}^{(2)}$ , or

$$E_v U_{\Delta} M_v \psi = U_{\Delta} M_v \psi.$$

By the stability considerations (Lemma 2 below, with  $X_1 = E_v$  to guarantee  $p_v > 0$ , and  $Y = E_v U_{\Delta} M_v, Z = U_{\Delta} M_v$ ),

$$E_v (U_{\Delta} M_v) = (U_{\Delta} M_v).$$

Since

$$E_v = M_v^{\dagger} M_v = M_v^{\dagger} U_{\Delta}^{\dagger} U_{\Delta} M_v = (U_{\Delta} M_v)^{\dagger} (U_{\Delta} M_v),$$

we can apply Lemma 4 below (with  $E_v = E$  and  $U_{\Delta} M_v = S$ ) to establish that all eigenvalues of  $E_v$  are 0's and 1's. Therefore  $E_v$  is an orthogonal projector operator. A POVM  $(E_0, E_1)$  with both components orthogonal projectors is a conventional observable.  $\square$

**Lemma 1.** *For a unitary operator  $U$  and an observable  $A$  with eigenprojectors  $P_0, P_1$ , if  $U$  and  $A$  commute then  $U^{\dagger} P_v U = P_v$  for  $v = 0$  and  $v = 1$ ; and if  $U^{\dagger} P_v U = P_v$  for either  $v = 0$  or  $v = 1$ , then  $U$  and  $A$  commute.*

*Proof.*  $U$  and  $A$  commute if and only if  $U$  commutes with  $P_1$  (associated with eigenvalue 1), because  $A = P_1$ . We should prove therefore that

$$U P_1 = P_1 U \iff U^{\dagger} P_1 U = P_1 \iff U^{\dagger} P_0 U = P_0.$$

For the first equivalence, if  $U P_1 = P_1 U$ , then  $U^{\dagger} P_1 U = U^{\dagger} U P_1 = P_1$ ; conversely, if  $U^{\dagger} P_1 U = P_1$ , then  $U P_1 = U U^{\dagger} P_1 U = P_1 U$ . For the second equivalence, if  $U^{\dagger} P_v U = P_v$ , then  $U^{\dagger} P_{1-v} U = U^{\dagger} (I - P_v) U = I - U^{\dagger} P_v U = I - P_v = P_{1-v}$ .  $\square$

**Lemma 2.** *Let  $X_1, \dots, X_n, Y, Z$  be some matrices. The statement*

$$\langle X_1 \psi, \psi \rangle > 0, \dots, \langle X_n \psi, \psi \rangle > 0$$

$$\Downarrow$$

$$Y \psi = Z \psi$$

*holds stable for  $\psi$  if and only if  $Y = Z$ .*

*Proof.* The “if” part is trivial. For the “only if” part, by the definition of stability the initial state can be chosen as

$$\overline{\psi} = \frac{\psi + \delta}{\|\psi + \delta\|},$$

where  $\delta \in B_r(0)$  (open ball of radius  $r$  centered at 0). By continuity considerations,  $r$  can be chosen sufficiently small for  $\langle X_1 \psi, \psi \rangle, \dots, \langle X_n \psi, \psi \rangle$  to remain positive. But  $Y \overline{\psi} = Z \overline{\psi}$  any such  $\overline{\psi}$ , and we have

$$Y \frac{\psi + \delta}{\|\psi + \delta\|} = Z \frac{\psi + \delta}{\|\psi + \delta\|},$$



whence

$$Y\delta = Z\delta.$$

Since every vector is collinear to some  $\delta$ ,  $Y$  and  $Z$  coincide.  $\square$

**Lemma 3.** *Let  $(E_1, E_2)$  be a POVM. If  $\langle E_v \phi, \phi \rangle = 1$  and  $\|\phi\| = 1$ , then  $E_v \phi = \phi$ .*

*Proof.* Writing  $E_v \phi = c\phi + \gamma$ , with  $\gamma \perp \phi$ , we see that  $\langle E_v \phi, \phi \rangle = 1$  implies  $c = 1$ . Choose an orthonormal basis  $(e_1, \dots, e_n)$  in the Hilbert space so that  $\phi = e_1$ . In this basis  $\gamma = \sum_{i=2}^n u_i e_i$ . Assume that  $\gamma \neq 0$ , and let, with no loss of generality,  $u_2 \neq 0$ . The components  $E_v$  in this basis is

$$E_v = \begin{pmatrix} 1 & \overline{u_2} & \dots & \overline{u_n} \\ u_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ u_n & \dots & \dots & \dots \end{pmatrix},$$

because when multiplied by  $\phi = e_1 = (1, 0, \dots, 0)^\top$ , it should yield  $e_1 + \sum_{i=2}^n u_i e_i = \phi + \gamma$ . Then the other component in this basis is

$$E_{1-v} = \begin{pmatrix} 0 & -\overline{u_2} & \dots & -\overline{u_n} \\ -u_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ -u'_n & \dots & \dots & \dots \end{pmatrix},$$

because  $E_v + E_{1-v} = I$ . But with  $u_2 \neq 0$ , the leading principal minor

$$\begin{vmatrix} 0 & -\overline{u_2} \\ -u_2 & \dots \end{vmatrix} < 0,$$

which contradicts the requirement that  $E_{1-v}$  be positive semidefinite. This contradiction shows that  $\gamma = 0$ .  $\square$

**Lemma 4.** *Let  $E = S^\dagger S$  be a component of a POVM, and let  $ES = S$ . Then all eigenvalues of  $E$  are 0's and 1's.*

*Proof.* Since  $E$  is Hermitian, we can select a basis consisting of its eigenvectors. In this basis matrix  $E$  is diagonal with the diagonal elements  $\lambda_1, \dots, \lambda_n$  (the eigenvalues of  $E$ ). Suppose that one of these elements, say  $\lambda_1$ , is not 1. From  $ES = S$  it follows that  $\lambda_1 s_1 = s_1$ , where  $s_1$  is the first row of  $S$ . Therefore  $s_1$  of  $S$  consists of zeros. But since  $E = S^\dagger S$ , we have  $\lambda_1 = \langle s_1, s_1 \rangle = 0$ . This proves that  $\lambda_1, \dots, \lambda_n$  consists of 0's and 1's.  $\square$